Simulation of a Networked Heterogeneous Van der Pol Oscillators

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I. Simulation result

A. Heterogeneous oscillators

A network of four Van der Pol oscillators are considered. The diffusive coupling law is used with $a = b = 1$, where the graph Laplacian is given as

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

The parameters c_i, w_i and the initial conditions $\mathsf{x}_i(0)$, $\mathsf{v}_i(0)$ are listed in the following table.

TABLE I
Parameters of Van der Pol oscillators

	c_i	w_i	$\mathsf{x}_i(0)$	$\mathsf{v}_i(0)$
Agent 1	0.9	0.9	-1	1
Agent 2	1.2	3.2	0.6	-0.4
Agent 3	0.8	2.8	1	0.7
Agent 4	1.1	4.1	-0.3	-1

Note that the blended dynamics of Van der Pol oscillators has its parameters $\hat{w} := \sqrt{\frac{\sum_{i=1}^N w_i^2}{N}} = 2.9875$ and $\hat{c} := \frac{\sum_{i=1}^N c_i w_i}{N\hat{w}} = 0.954$. Fig. 1. (a-c) illustrates the trajectory of agent 4 on the x_i-v_i plane with the red lines, where the blue line indicates the limit cycle. Three cases are different only in the coupling gain used, where its values are given as (a) 20, (b) 50, and (c) 100. We observe that for each coupling gain k, the solution trajectory converges to some limit cycle. The limit cycles are close, and it gets closer as k increases. Even though the limit cycle is heterogeneous for agents (for each k), we can still observe practical phase synchronization from Fig. 1. (d). Fig. 1. (d) is a graph representing the value $\sum_{i\neq j} \|\mathsf{x}_i(t) - \mathsf{x}_j(t)\|$ (y-axis) with respect to the time (x-axis), where the color indicates the difference of coupling gain (Black: $k = 20$, Blue: $k = 50$, Red: $k = 100$). It is seen that the value in the graph adopts limit supremum (for each k), and thus, oscillators are phase cohesive (for each k). Since limit supremum decreases to zero as gain increases, we can observe practical phase synchronization.

B. Homogeneous oscillators

To make clear the difference with the earlier subsection, we consider a network of identical Van der Pol oscillators having their parameters $c_i = \hat{c} = 0.954$ and $w_i = \hat{w} = 2.9875$. Fig. 2. (a-c) illustrates the trajectory of agent 2 on the x_i-v_i plane with the red lines, where the blue line indicates the limit cycle of the blended dynamics. Three cases are different only in the coupling gain used, where its values are given as (a) 0.1, (b) 0.5, and (c) 5. The behavior in Fig. 2. (a-c) is similar to that of Fig. 1. (a-c) when coupling gain is sufficiently large (that is $k \geq 0.5$). However, the limit cycle of each agent (for each k) is identical to the limit cycle of the blended dynamics. Such property can be observed in Fig. 2. (b-c) as a convergence of the trajectory to the limit cycle of the blended dynamics, and in Fig. 2. (d) as a convergence of $\sum_{i\neq j} \|\mathsf{x}_i(t) - \mathsf{x}_j(t)\|$ to zero (Black: $k = 0.1$, Blue: $k = 0.5$, Red: $k = 5$). Finally, note that Fig. 2. (d) also shows that the oscillators achieve exact synchronization.

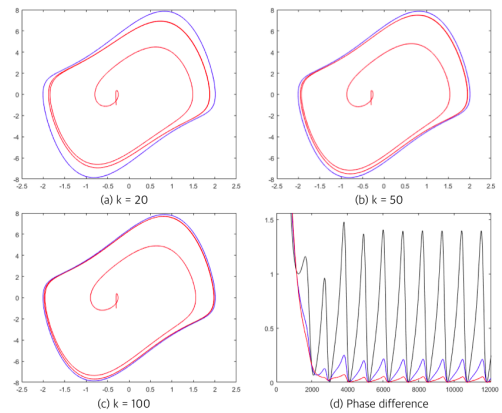

Fig. 1. (a-c) Trajectory of agent 4 with $k = 20, 50, 100$ (red) and the limit cycle of the blended dynamics (blue) (d) Evolution of phase difference $\sum_{i\neq j} \|\mathsf{x}_i(t) - \mathsf{x}_j(t)\|$.

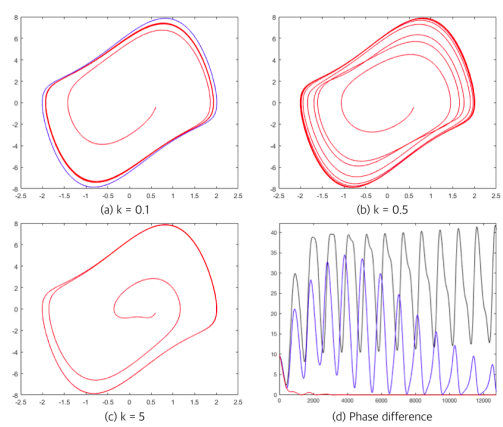

Fig. 2. (a-c) Trajectory of agent 2 with $k = 0.1, 0.5, 5$ (red) and the limit cycle of the blended dynamics (blue) (d) Evolution of phase difference $\sum_{i\neq j} \|\mathsf{x}_i(t) - \mathsf{x}_j(t)\|$.